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Note

The complexity of some acyclic improper colourings

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ABSTRACT

In this paper we will consider acyclic bipartition of the vertices of graphs, where acyclic means that the edges whose endpoints are in different parts of the partition induce a forest. We will require that the vertices belonging to the same partition induce graphs from particular class. We will search for acyclic bipartitions of cubic and subcubic graphs.

1. Introduction

We consider only finite simple graphs. For a graph G we denote its vertex set by $V(G)$ and its edge set by $E(G)$. The set of neighbours of a vertex $v \in V(G)$ is denoted by $N_G(v)$ or $N(v)$, and $\Delta(G)$ denotes the maximum degree of G. Furthermore, for any $k \geq 0$, we denote by $\mathcal{S}_k$ the class of all graphs having maximum degree at most k. We distinguish the class of edgeless graphs (equivalent to $\mathcal{S}_0$) and denote it by $\mathcal{O}$. We say that a graph G is *subcubic* if $G \in \mathcal{S}_3$ and *cubic* if G is 3-regular. Let $\mathcal{D}_1$ denote the class of all acyclic graphs. A graph is a *linear forest* if it is acyclic and has maximum degree at most 2; let $\mathcal{LF} = \mathcal{D}_1 \cap \mathcal{S}_2$. For all undefined concepts we refer the reader to [4].

A *k-colouring* of a graph G is a mapping c from the set of vertices of G to the set $\{1, \dots, k\}$ of colours. We can also regard a k-colouring of G as a partition of the set $V(G)$ into *colour classes* $V_1, \dots, V_k$ such that each V_i is the set of vertices with colour i. In many situations it is desired that the particular set V_i has some particular properties. Let $\mathcal{P}_1, \dots, \mathcal{P}_k$ be nonempty classes of graphs. A k-colouring of a graph G is called a *$(\mathcal{P}_1, \dots, \mathcal{P}_k)$-colouring* of G if for $1 \leq i \leq k$ the subgraph induced in G by the colour class V_i belongs to $\mathcal{P}_i$. Such a colouring is called an *acyclic $(\mathcal{P}_1, \dots, \mathcal{P}_k)$-colouring* if for every two distinct colours i and j the subgraph formed by the edges whose endpoints have colours i and j is acyclic. In other words, every 2-coloured cycle in G contains at least one monochromatic edge. A bichromatic cycle (path) having no monochromatic edge will be called an *alternating cycle* (*path*). When G has a $(\mathcal{P}_1, \dots, \mathcal{P}_k)$-colouring, we write $G \in \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_k$. When G has an acyclic $(\mathcal{P}_1, \dots, \mathcal{P}_k)$-colouring, we write $G \in \mathcal{P}_1 \odot \cdots \odot \mathcal{P}_k$.

An acyclic $(\mathcal{P}_1, \dots, \mathcal{P}_k)$-colouring is called an *acyclic k-colouring* if for $1 \leq i \leq k$ the class $\mathcal{P}_i$ is the set of all edgeless graphs. The minimum k such that G has an acyclic k-colouring is called the *acyclic chromatic number* of G, denoted by $\chi_a(G)$.

The concept of acyclic vertex colouring was introduced by Grünbaum in [7]. He showed that if $G \in \mathcal{S}_3$, then $\chi_a(G) \leq 4$. In [11] an $O(n)$-time algorithm that uses four colours to acyclically colour the vertices of any subcubic graph is given. On the other hand, Kostochka proved in [8] that it is an NP-complete problem to decide for a given graph G whether $\chi_a(G) \leq 3$.

The complexity of $(\mathcal{P}_1, \mathcal{P}_2)$-colouring and acyclic $(\mathcal{P}_1, \mathcal{P}_2)$-colouring were also investigated. In [9] it was proved that the problem of $(\mathcal{O}, \mathcal{S}_1)$-colouring of cubic graphs is NP-complete. We will show in Section 2 that also the problem of an acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring of cubic graphs is NP-complete.

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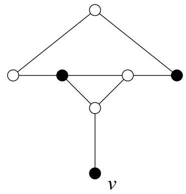

Fig. 1. The graph Q and its acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring (black vertices have colour 1 and white vertices have colour 2).

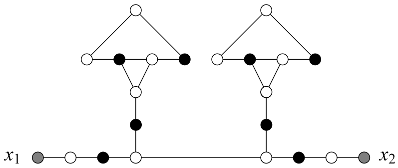

Fig. 2. The graph Q'. The grey vertices can be coloured with 1 or 2.

It is known that every subcubic graph has an $(\mathcal{S}_1, \mathcal{S}_1)$-colouring. Moreover, any 2-colouring of G with the maximum number of 2-coloured edges is an $(\mathcal{S}_1, \mathcal{S}_1)$-colouring of G. In [5] the authors proved the following theorem.

Theorem 1 ([5]). *Every cubic graph can be $(\mathcal{S}_1, \mathcal{S}_1)$-coloured in linear time.*

Boiron et al. [2] found infinitely many cubic graphs that have no acyclic $(\mathcal{S}_1, \mathcal{S}_1)$-colouring. The question was whether we can easily recognize such graphs. We prove that it is an NP-complete problem to decide for a cubic graph if it is acyclically $(\mathcal{S}_1, \mathcal{S}_1)$-colourable.

In [12] the problem of $(\mathcal{O}, \mathcal{D}_1)$-colouring was investigated. The authors proved that it is an NP-complete problem to decide for a graph with maximum degree 4 whether it is $(\mathcal{O}, \mathcal{D}_1)$-colourable, and they showed that every subcubic graph other than K_4 has an $(\mathcal{O}, \mathcal{D}_1)$-colouring. Since there exist infinitely many graphs with maximum degree 3 and acyclic chromatic number 4, we have $\mathcal{S}_3 \nsubseteq \mathcal{O} \odot \mathcal{D}_1$. In [2] it was proved that every subcubic graph has an $(\mathcal{S}_2, \mathcal{D}_1)$-colouring and $\mathcal{S}_3 \setminus \{K_4, K_{3,3}\} \subseteq \mathcal{D}_1 \odot \mathcal{D}_1$. Borowiecki et al. [3] and Addario-Berry et al. [1] independently proved that every subcubic graph has an acyclic $(\mathcal{S}_2, \mathcal{S}_2)$-colouring.

In Section 3 we prove that $\mathcal{S}_3 \setminus \{K_4, K_{3,3}\} \subseteq \mathcal{D}_1 \odot \mathcal{LF}$ and give a polynomial-time algorithm that provides an acyclic $(\mathcal{D}_1, \mathcal{LF})$-colouring of cubic graphs other than K_4 and $K_{3,3}$. This result improves that given in [2].

2. NP-complete problems

In this section we will show that the problem of deciding whether a given graph $G \in \mathcal{S}_3$ is acyclically $(\mathcal{O}, \mathcal{S}_1)$-colourable is NP-complete. Let the vertices with colour 1 induce a graph that belongs to $\mathcal{O}$ and the vertices with colour 2 induce a graph that belongs to $\mathcal{S}_1$. First we will present some special graphs and their properties.

Observation 1. *Let Q be the graph depicted in Fig. 1. In every acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring of Q the vertex v has colour* 1.

The graph Q' depicted in Fig. 2 contains two disjoint copies of Q as induced subgraphs. Its acyclic $(\mathcal{O}, \mathcal{S}_1)$-colourings (see Fig. 2) are almost entirely determined by these subgraphs.

The most important property of the graph Q' is the following.

Observation 2. *In every acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring of Q' there is no alternating path between the vertices x_1 and x_2.*

Consider now the graph W depicted in Fig. 3.

Observation 3. *In every acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring of W, the vertices f and g must have colour* 2.

The above fact leads us to remark that the graph W has only two possible acyclic $(\mathcal{O}, \mathcal{S}_1)$-colourings.

Observation 4. *In every acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring of the graph W the vertices a, e and c have the same colour.*

Unfortunately in each of the two colourings there is an alternating path joining the vertices a and e, which would be an undesirable situation in the construction used in the proof of our first theorem. Because of this we make some modifications which will be based on the properties of the graph Q'. So let us present the graph W'.

We can see that Observations 3 and 4 are still true for this graph. From Observation 2 follows the next one.

Observation 5. *Let W' be the graph depicted in Fig. 4. In every acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring of the graph W' there is no alternating path joining the vertices a and e.*

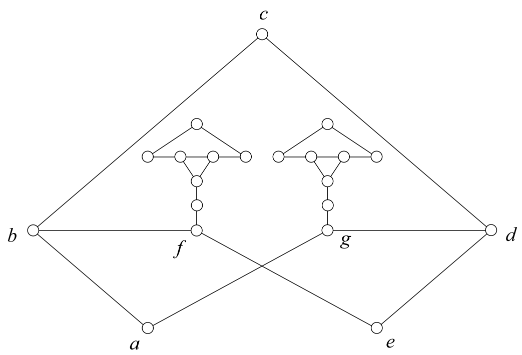

Fig. 3. The graph W.

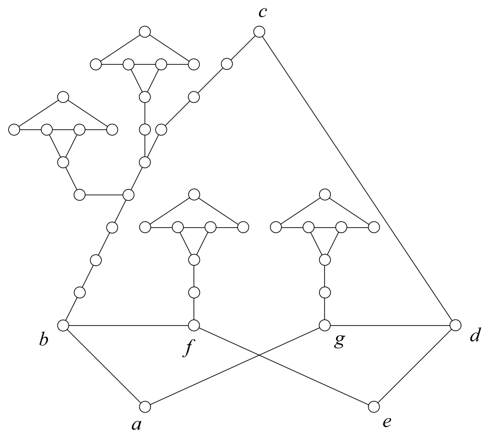

Fig. 4. The graph W'.

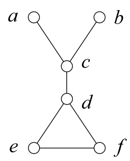

Fig. 5. The graph F.

Finally we present the graph F depicted in Fig. 5. We are interested in its special colourings, where the colours of the vertices a and b are fixed.

Observation 6. *In an $(\mathcal{O}, \mathcal{S}_1)$-colouring of the graph F, if vertices a and b have the same colour, then d also has that colour.*

In our NP-completeness proof we will use the MONOTONE ONE-IN-THREE 3SAT (M 1-IN-3 3SAT) problem. Let $\mathcal{C}$ be a collection of m clauses over the set $\mathcal{U}$ such that each clause $C \in \mathcal{C}$ has exactly three unnegated literals. We ask whether there exists a truth assignment such that each clause has exactly one true literal. It is known that this problem is NP-complete; see [6,10] for details. Clearly the problem of acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring is in NP. We will show that it is NP-complete.

Theorem 2. *Recognizing whether a given graph G has an acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring is NP-complete, even for graphs with maximum degree* 3.

Proof. We will make the polynomial-time reduction from M 1-IN-3 3SAT to the problem of recognizing graphs that are acyclically $(\mathcal{O}, \mathcal{S}_1)$-colourable. Let $\mathcal{C}$ denote the set of m clauses with the set of literals $X = \{x_1, \ldots, x_n\}$. For each clause $C_b = \{x_s, x_t, x_p\} \in \mathcal{C}$, we construct a graph G_{C_b} that is a triangle with vertex set $\{x_{s_b}, x_{t_b}, x_{p_b}\}$. We will call the vertex x_{s_b} a *corresponding vertex* of x_s, and similarly for x_t and x_p. Let X_{x_s} denote the set of all corresponding vertices of the literal x_s.

For each literal x_i with $1 \leq i \leq n$, we construct the graph G_{x_i} in the following way. Make $4k$ copies of the graph W', where k is the number of clauses containing x_i. To each labelled vertex in these graphs (see Fig. 4) we add a subindex j denoting the number of the copy. Also, we add edges joining the vertices e_j and a_{j+1} for $1 \leq j \leq 4k - 1$. From Observations 4 and 3,

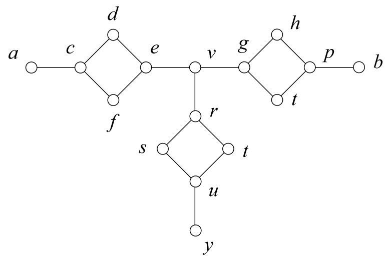

Fig. 6. The graph H.

it follows that in the resulting graph all the vertices c_j where j is odd have the same colour. The same remark is true for the vertices c_j where j is even. Let $P = \{p_1, \dots, p_k\}$ be the set of pairs $p_t = \{c_{4(t-1)+1}, c_{4(t-1)+3}\}$, $1 \le t \le k$.

For each graph G_{x_i} we next add vertices $y_1, \dots, y_k$. Let Y_{x_i} denote the set $\{y_1, \dots, y_k\}$ in the graph G_{x_i}. For each t, we make y_t adjacent to the vertices in the pair p_t. One can see that all acyclic $(\mathcal{O}, \mathcal{S}_1)$-colourings of G_{x_i} give the same colour to all vertices in Y_{x_i}. Next we connect the resulting graphs. For each literal x_i we add a perfect matching linking X_{x_i} to Y_{x_i}. We denote by G the so-obtained graph.

Suppose now that our graph G has an acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring. In every graph G_{C_b} exactly one vertex has colour 1. Moreover, from the properties of the graphs G_{x_i} and Observation 6, we obtain that the corresponding vertices of every literal x_i have the same colour. So clearly the assignment in which each literal whose corresponding vertices have colour 1 is set TRUE and the others are set FALSE is satisfying assignment for the given instance of the M 1-IN-3 3SAT problem.

Assume that the given instance of the M 1-IN-3 3SAT problem is satisfiable. Let us colour the corresponding vertices of the literal x_i in the following way. If x_i has a true value then we use colour 1, otherwise we use colour 2. From the previous considerations we know that this colouring can be properly extended so that every vertex of the graph G is acyclically $(\mathcal{O}, \mathcal{S}_1)$-coloured. $\square$

It is a simple matter to see that, except for C_4, every graph in $\mathcal{S}_2$ has an acyclic $(\mathcal{O}, \mathcal{S}_1)$-colouring.

Next we consider the complexity of acyclic $(\mathcal{S}_1, \mathcal{S}_1)$-colouring. We will first introduce a graph that will play the main role in our next proof. To distinguish to which colour class the particular vertex belongs we will use colours 1 and 2.

Let us consider the graph H presented in Fig. 6.

One can see that the following observation is true.

Observation 7. *In every acyclic $(\mathcal{S}_1, \mathcal{S}_1)$-colouring of the graph H the vertices a, y and b have the same colour.*

In proving the next theorem we will use the graph H and another version of the 3SAT problem. It is called MONOTONE NOT-ALL-EQUAL 3SAT (MNAE-3SAT) and is known to be NP-complete [6,10]. In this problem we ask whether there exists a truth assignment such that each clause has at least one true literal and at least one false literal. Moreover, all literals in the clauses are unnegated.

Theorem 3. *Recognizing whether a given graph G has an acyclic $(\mathcal{S}_1, \mathcal{S}_1)$-colouring is NP-complete, even for graphs with maximum degree* 3.

Proof. Suppose that we are given an instance of MNAE-3SAT where $\mathcal{C}$ is a set of m clauses with the set of literals $X = \{x_1, \dots, x_n\}$. We will make a polynomial-time reduction from this problem to the problem of recognizing acyclically $(\mathcal{S}_1, \mathcal{S}_1)$-colourable graphs. For $C_j = \{x_s, x_t, x_p\} \in \mathcal{C}$ we construct a graph G_{C_j} that is a triangle with vertex set $\{x_{s_j}, x_{t_j}, x_{p_j}\}$. As before, we call the vertex x_{s_j} a *corresponding vertex* of x_s, and similarly for x_t and x_p.

For each literal x_i with $1 \le i \le n$, construct a graph G_{x_i} in the following way. Make k copies of the graph H, where k is the number of clauses containing x_i. We modify the vertex labels on H in Fig. 6 by adding the subindex l denoting the number of the copy of the graph H. Then we identify the vertices b_l and a_{l+1} for $1 \le l \le k-1$. In every acyclic $(\mathcal{S}_1, \mathcal{S}_1)$-colouring of the graph G_{x_i} the vertices $b_1, \dots, b_k$ have the same colour and there is no alternating path joining any two vertices in $\{y_1, \dots, y_k\}$. Let $Y_{x_i} = \{y_1, \dots, y_k\} \subset V(G_{x_i})$. Now we connect the graphs G_{C_j} and G_{x_i}, for $1 \le j \le m$ and $1 \le i \le n$.

For every literal x_i, let X_{x_i} denote the set of the corresponding vertices of x_i; note that $|X_{x_i}| = k$. We identify vertex $y_r \in Y_{x_i}$ with exactly one vertex of X_{x_i}, for $1 \le r \le k$ according to some one-to-one mapping between X_{x_i} and Y_{x_i}. Now the instance of our problem is complete; let G denote the resulting graph.

Let us suppose that there is a truth assignment such that each clause has at least one true and one false literal. This assumption allows us to properly colour the graph G in the following way. If the literal x_i is true, then we colour all its corresponding vertices with colour 1. Otherwise we colour them with colour 2. From Observation 7 and the properties of the graphs G_{x_i}, it follows that this colouring extends to an acyclic $(\mathcal{S}_1, \mathcal{S}_1)$-colouring of G.

Now we suppose that the graph G has an acyclic $(\mathcal{S}_1, \mathcal{S}_1)$-colouring. In such a colouring, each subgraph G_{C_j} has at least one vertex of each colour. Furthermore, by Observation 7 we know that all the corresponding vertices of the same literal have the same colour. That implies that if we assign the truth value to all the vertices whose corresponding vertices have colour 1 and the false value otherwise, then we have a satisfying assignment for MNAE-3SAT. $\square$

3. Positive results

Boiron et al. in [2] investigated 2-colourings of subcubic graphs. The authors asked whether almost every subcubic graph has an acyclic $(\mathcal{D}_1, \mathcal{S}_1)$-colouring. They pointed out three graphs not having an acyclic $(\mathcal{D}_1, \mathcal{S}_1)$-colouring. They also showed that $\mathcal{S}_3 \setminus \{K_4, K_{3,3}\} \subseteq \mathcal{D}_1 \odot \mathcal{D}_1$ and $\mathcal{S}_3 \subseteq \mathcal{D}_1 \odot \mathcal{S}_2$. Using similar methods we will improve these two results by proving the following theorem:

Theorem 4. $\mathcal{S}_3 \setminus \{K_4, K_{3,3}\} \subseteq \mathcal{D}_1 \odot \mathcal{LF}$.

Proof. Since every graph from $\mathcal{S}_3$ is a subgraph of a cubic graph, it suffices to prove this statement for cubic graphs. By Theorem 1 every cubic graph has an $(\mathcal{S}_1, \mathcal{S}_1)$-colouring, so it also has a $(\mathcal{D}_1, \mathcal{LF})$-colouring. Let G be a 3-regular graph and c a $(\mathcal{D}_1, \mathcal{LF})$-colouring of G. Assume that vertices with colour 1 induce a forest and vertices with colour 2 induce a linear forest. Let $C = (x_0, x_1, \ldots, x_{2p-1})$ be an alternating cycle of G and let y_i be the neighbour of x_i distinct from x_{i-1} and x_{i+1} (indices modulo $2p$). We will modify the colouring c in such a way that C is no longer an alternating cycle, that no new alternating cycles appear, and that the graph G is still $(\mathcal{D}_1, \mathcal{LF})$-coloured. By repeating such a recolouring we decrease the number of alternating cycles, and we eventually obtain an acyclic $(\mathcal{D}_1, \mathcal{LF})$-colouring of G. Let us consider three cases:

Case 1. There exists i $(0 \le i \le 2p-1)$ such that $c(y_i) = c(y_{i+1})$.

If x_i and y_i have the same colour, then we change the colour of x_i; otherwise, we change the colour of x_{i+1}. Since the vertex whose colour was changed now has two neighbours that are coloured with its colour, no new alternating cycle appears. The graph is still $(\mathcal{D}_1, \mathcal{LF})$-coloured.

Case 2. For every i for $0 \le i \le 2p-1$, $c(y_i) \neq c(y_{i+1})$ (indices modulo $2p$) and $c(x_i) \neq c(y_i)$.

Let x_j be any vertex with colour 2, then it suffices to change the colour of x_j.

Case 3. For every i for $0 \le i \le 2n-1$, $c(y_i) \neq c(y_{i+1})$ (indices modulo $2p$) and $c(x_i) = c(y_i)$.

Subcase 3.1. If there exists i such that x_i and x_{i+2} are in distinct components of the subgraph induced by vertices coloured with their colour, then it suffices to change the colour of x_{i+1}.

Subcase 3.2. Suppose that there exists vertex y_i with colour 1 having three neighbours with colour 1. Then we change the colour of x_i and x_{i+1}. Now y_i has two neighbours with colour 1 and cannot be in any alternating cycle. Hence x_i and x_{i+1} also are not in any alternating cycle. Moreover, the new colouring is still a $(\mathcal{D}_1, \mathcal{LF})$-colouring of G.

Subcase 3.3. Suppose that there is some y_i that belongs to C. Since x_i and y_i belong to the same tree (induced by their colour) and $d(x_i) = d(y_i) = 3$, it follows that the length of C is exactly 4 (since we are not in subcase 3.1). Since the graph G is not K_4, it suffices to change the colours of x_i and x_{i+1}.

In the remaining cases we may thus assume that

(i) the vertices $\{x_0, x_2, \ldots, x_{2p-2}\}$ are coloured with 1 and belong to the same tree induced by vertices with colour 1 and the vertices $\{x_1, x_3, \ldots, x_{2p-1}\}$ are coloured with 2 and belong to the same linear tree induced by vertices with colour 2;

(ii) every y_i has exactly two neighbours having colour $c(y_i)$ and one neighbour having the other colour;

(iii) $y_i \notin V(C)$ for $0 \le i \le 2p-1$.

Subcase 3.4. Suppose that there exists y_i such that $c(y_i) = 2$ and x_i is the only neighbour of y_i in C (i.e. $N(y_i) \cap V(C) = \{x_i\}$). We change the colour of x_{i-1} and y_i. Since x_{i-1} has two neighbours coloured with 2, it follows that there is no alternating cycle that contains x_{i-1}. Since x_i is not in any monochromatic cycle, we have that also x_{i-1} is not in any monochromatic cycle. The vertex x_{i-1} and its neighbours with colour 2 have at least one neighbour with colour 1, thus the vertices of colour 1 still induce a linear forest. The vertex y_i has two neighbours with colour 2, so no monochromatic cycle contains y_i. Any alternating cycle containing y_i would contain x_i and consecutive vertices of C. All vertices of C except x_i have neighbours outside C that are coloured with their colour, then they cannot belong to any alternating cycle.

Subcase 3.5. Suppose that there exists y_i such that $c(y_i) = 1$ and x_i is the only neighbour of y_i in C. If there is no alternating path from y_i to y_{i+1}, then we change the colour of x_i and x_{i+1} (we obtain a $(\mathcal{D}_1, \mathcal{LF})$-colouring with fewer alternating cycles). Otherwise we change the colour of y_i and x_{i+1}. The vertex x_{i+1} has two neighbours with colour 1, so it is not contained in any alternating cycle. Since x_i is not contained in any monochromatic cycle, also x_{i+1} is not contained in any monochromatic cycle. By (ii) the vertex y_i has two neighbours having colour 1. Hence no monochromatic cycle contains y_i. There is also no alternating cycle containing x_i because it would have to go through vertices of C. The only neighbour of y_i having colour 2 was in the alternating path. Thus it has at least one neighbour coloured with 1 and the vertices with colour 2 induce a linear forest.

Subcase 3.6. We still have one case to consider: every y_i has at least two neighbours in C. From (i) and (ii) it follows that C has exactly 4 vertices and $y_0 = y_2, y_1 = y_3$ (see Fig. 7). Since G is not $K_{3,3}$, vertices y_0 and y_1 are not adjacent. By (ii) y_0 has a neighbour outside the cycle having colour 2, and y_1 has a neighbour outside the cycle having colour 1. We change the colour of vertices x_0, x_1, y_1. Now C is not an alternating cycle. Since y_0 is not in any monochromatic cycle, vertices x_1, x_2, y_1 also cannot be in any monochromatic cycle. Each of x_1, x_2, y_1 has two neighbours with colour 1, so they are not contained in any alternating cycle. The vertex x_0 has only one neighbour with colour 2, so it is not contained in any monochromatic cycle. Furthermore, there is no alternating cycle containing x_0, because x_1 is not in any alternating cycle. □

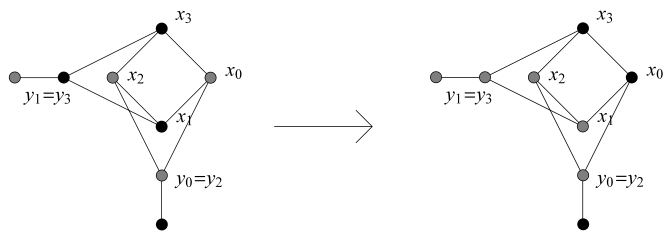

Fig. 7. Illustration for subcase 3.6.

Finally, we discuss the running time of the algorithm described in the proof of Theorem 4. In the algorithm we first find a $(\mathcal{D}_1, \mathcal{LF})$-colouring of G. Then for every vertex v we look for an alternating cycle C containing v. If such an alternating cycle exists, then we destroy it using the procedure described in the proof of Theorem 4. By Theorem 1 every cubic graph can be $(\mathcal{D}_1, \mathcal{LF})$-coloured in linear time. In [3] an $O(n)$-time algorithm that finds an alternating cycle C containing v or gives a message that such a cycle does not exist is given. We can also use this algorithm for finding an alternating path in subcase 3.5 of the proof of Theorem 4. Thus there is an $O(n)$-time algorithm that destroys the alternating cycle C in such a way that a new alternating cycle does not appear and the graph G is still $(\mathcal{D}_1, \mathcal{LF})$-coloured. The time complexity of the algorithm is thus $O(n^2)$, and we obtain the following theorem.

Theorem 5. *Every cubic graph except K_4 and $K_{3,3}$ can be acyclically $(\mathcal{D}_1, \mathcal{LF})$-coloured in polynomial time.*

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